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LONGITUDINAL DIFFUSION IN A FLOW THROUGH A TUBE

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Abstract—The results of research concerned with a fluid mixing during the movement in a tube, are given. A method of definining the one-dimension theory of matter transfer, accounting for the difference of mixture component velocities is presented. The longitudinal transfer in a zone of "passive" fluids contact is discussed in detail.

It has been possible to formulate the theory, which generalises the well-known Taylor and Aris models. The theory presented is based on the integro-differential equation, accounting for the delay effects. It has been possible to describe the experimental facts, which had no explanation so far, in bounds of the given theory.

1. INTRODUCTION

The theory of longitudinal matter transfer in a flow through a tube (Taylor 1953, 1954) is based on the diffusion equation with some virtual coefficient different from the molecular coefficient.

It follows from this theory that dispersion is growing with time by the linear law. But the experimental data (Bailey & Gogarty 1962) show that for time spans $\tau \leqslant 1$ dispersion is growing much more slowly than in the theory.

An attempt to correlate the theory and the experiment for $\tau \leqslant 1$ by including a virtual coefficient dependence on time has been presented by Gill & Sankarasubramanian (1970).

Such an approach is not satisfactory from the viewpoint of model construction methods. In Taylor's theory the virtual coefficient depends on parameters (U_0, R, D) and results in the universality of the model. Including the time-dependent coefficient in the model limits its application to some unknown category of problems for which the adopted dependence is true. The experimental results cast doubt on Taylor's model, based on the diffusive transfer equation.

2. ONE-DIMENSION EQUATION OF MATTER TRANSFER

During fluid mixing in a tube local convective and turbulent diffusion processes form a mixture zone, in which the fluid concentration is smoothly changing in length and is almost constant in the flow cross-section. This is the reason for using mean concentrations instead of local ones for describing the mixing process.

Each of the mixing fluids is assumed to fill the whole area of joint flow continuously, that is, it presents a continuum with its own density ρ_i ($i = 1, 2$) and mean tube cross-section velocity v_i ($i = 1, 2$).

It has been shown by experiment that in mixing fluids with volumes V_1 and V_2 a mixture is obtained the volume of which is fairly accurately equal to the sum of component volumes, i.e.

$$V_s = V_1 + V_2. \qquad [1]$$

It will be noted that the property of volume keeping in mixing does not take place for all fluids.

Volume concentrations θ_1 and θ_2 are introduced for each component according to formulae

$$\theta_1 = \frac{V_1}{V_S} \quad \text{and} \quad \theta_2 = \frac{V_2}{V_S}. \qquad [2]$$

It follows from [1] that

$$\theta_1 + \theta_2 = 1. \qquad [3]$$

Using volume concentrations and true densities of fluids it is possible to calculate the reduced densities $\rho_1{}^0$ and $\rho_2{}^0$. We get

$$\rho_1{}^0 = \frac{\rho_1 V_1}{V_S} = \rho_1 \cdot \theta_1, \quad \rho_2{}^0 = \frac{\rho_2 V_2}{V_S} = \rho_2 \cdot \theta_2. \qquad [4]$$

Mass conservation equations for each component are

$$\frac{\partial \rho_i{}^0}{\partial t} + \frac{\partial \rho_i{}^0 v_i}{\partial x} = 0, \quad (i = 1, 2). \qquad [5]$$

Using [4] and [5] the equations for the volume concentrations of mixture components are written as

$$\frac{\partial \theta_i}{\partial t} + \frac{\partial \theta_i v_i}{\partial x} = 0, \quad (i = 1, 2). \qquad [6]$$

Adding the L.H.S. of these equations and taking into consideration[3] we get

$$\frac{\partial}{\partial x}(\theta_1 v_1 + \theta_2 v_2) = 0. \qquad [7]$$

Hence it follows that

$$\theta_1 v_1 + \theta_2 v_2 = U_0. \qquad [8]$$

Velocity U_0 does not change along the tube and is equal to the mean velocity of particles in the uniform part of the flow. Equations [6] are written as follows

$$\frac{\partial \theta_i}{\partial t} + U_0 \frac{\partial \theta_i}{\partial x} = -\frac{\partial}{\partial x} J_i, \quad J_i = (v_i - U_0)\theta_i. \qquad [9]$$

Here J_i $(i = 1, 2)$ is the mean tube cross-section flow of matter of the i-th component in the movable coordinate system, moving with velocity U_0. It is easy to show that $J_1 = -J_2$.

3. FORMULAE FOR CALCULATING MATTER FLOWS

Values J_i are calculated by means of local matter flows. With this aim in view local volume concentrations C_i and velocity vectors of mixture components $\mathbf{W}_i$ are introduced.

The local flows consist of the convective and diffusive flows and are equal to

$$\mathbf{q}_i = \mathbf{W}_i C_i - \epsilon_i \nabla C_i, \qquad i = 1, 2. \qquad [10]$$

ϵ_i are the local matter-transfer coefficients. The projection of vector $\mathbf{q}_i$ on axis x is equal to

$$q_{xi} = u_i \cdot C_i - \epsilon_i \frac{\partial C_i}{\partial x}, \qquad u_i = W_{xi}, \qquad i = 1, 2. \qquad [11]$$

The same flow calculated in the movable coordinate system is equal to

$$q_{xi} = (u_i - U_0)C_i - \epsilon_i \frac{\partial C_i}{\partial x}, \qquad i = 1, 2. \qquad [12]$$

The mean tube cross-section value of this flow is equal to J_i. We get

$$J_i = \frac{1}{S}\iint_S q_{xi}\,\mathrm{d}\sigma = \frac{1}{S}\iint_S (u_i - U_0)C_i\,\mathrm{d}\sigma - \frac{1}{S}\iint_S \epsilon_i\frac{\partial C_i}{\partial x}\mathrm{d}\sigma, \qquad i = 1, 2. \qquad [13]$$

These formulae are used for calculating flows J_i in [9] when local concentrations C_i and profiles of longitudinal components of velocities u_i are known.

4. AN EQUATION FOR LOCAL CONCENTRATION

Let us consider simple conditions for two fluids flowing in a tube when the calculation of a matter flow is not difficult. The mixing fluids are taken to have identical densities and viscosities. In this case mixture density and viscosity are constant. It is supposed that the flow temperature is independent of the space coordinates (the temperature can change with time).

The assumption of the constancy of temperature and density of fluid particles in the mixing area enable us to exclude from our consideration the natural convection processes in the field of forces of gravity and to limit the study by the forced convection and the diffusive matter-transfer.

In the case under consideration the vector of movement velocity has only one component along the tube different from zero, which is the same for both mixture components, and the transfer coefficients ϵ_i are also equal. The diffusion equation for one of the mixture components is

$$\frac{\partial C}{\partial t} + u(r)\frac{\partial C}{\partial x} = \frac{1}{r^p}\frac{\partial}{\partial r}\left(r^p\epsilon(r)\frac{\partial C}{\partial r}\right) + \epsilon(r)\frac{\partial^2 C}{\partial x^2}, \qquad C = C(t, x, r). \qquad [14]$$

Here p is the symmetry factor. In a flat tube $p = 0$ and in a round tube $p = 1$.

The mass flow on the inner tube surface is equal to zero,

$$\left.\frac{\partial C}{\partial r}\right|_{r=R} = 0. \qquad [15]$$

The corresponding initial and boundary conditions on t and x should be added to the above conditions.

5. LAMINAR FLOW IN A ROUND TUBE

In this case the transfer coefficient $\epsilon(r)$ is constant and equal to the molecular diffusion coefficient D. Factor $p = 1$. Dimensionless variables of the following kind are introduced,

$$\tau = \frac{Dt}{R^2}, \qquad \xi = \frac{x}{R}, \qquad \eta = \frac{r}{R}. \qquad [16]$$

Using these variables, [14] and [15] are rewritten as follows

$$\frac{\partial C}{\partial \tau} + Y\Phi(\eta)\frac{\partial C}{\partial \xi} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial C}{\partial \eta}\right) + \frac{\partial^2 C}{\partial \xi^2},$$

$$Y = \frac{U_0 R}{D}, \quad C = C(\tau, \xi, \eta), \quad \tau > 0, \quad 0 < \eta < 1, \quad -\infty < a < \xi < b < \infty. \qquad [17]$$

$$\frac{\partial C}{\partial \eta}(\tau, \xi, 1) = \frac{\partial C}{\partial \eta}(\tau, \xi, 0) = 0. \qquad [18]$$

The initial condition is taken as

$$C(0,\xi,\eta)=\theta_0(\xi). \qquad [19]$$

This condition, in which the initial function depends on ξ only, is similar to the condition taken by Taylor and other authors.

One-dimension equation [9] is rewritten taking into account [13] (notation i is omitted):

$$\begin{gathered}\frac{\partial\theta}{\partial\tau}+Y\frac{\partial\theta}{\partial\xi}=2Y\frac{\partial}{\partial\xi}\int_0^1(\Phi(\eta)-1)\cdot C\cdot\eta\,\mathrm{d}\eta+\frac{\partial^2\theta}{\partial\xi^2},\\ \theta=\theta(\tau,\xi),\qquad \tau>0,\quad -\infty<a<\xi<b<+\infty.\end{gathered} \qquad [20]$$

Function $\Psi(\tau,\xi,\eta)$ equal to the difference between the local and mean concentrations $\Psi=C-\theta$ is introduced. Combining the addends of [17] and [20] we get the following equation for this function

$$\frac{\partial\Psi}{\partial\tau}-\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\Psi}{\partial\eta}\right)=2Y\frac{\partial}{\partial\xi}\int_0^1(\Phi(\eta)-1)\Psi\cdot\eta\,\mathrm{d}\eta-Y\frac{\partial\Psi}{\partial\xi}-Y(\Phi(\eta)-1)\frac{\partial C}{\partial\xi}+\frac{\partial^2\Psi}{\partial\xi^2}. \qquad [21]$$

5.1 *Solution of* [21]

This equation is solved by the successive approximation method when considered as a non-homogeneous equation with a free member in the R.H.S. As a null approximation we take $\Psi=0$. This approximation is substituted into the R.H.S. of the equation. We get

$$\frac{\partial\Psi}{\partial\tau}-\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\Psi}{\partial\eta}\right)=-Y(\Phi(\eta)-1)\frac{\partial\theta}{\partial\xi}. \qquad [22]$$

The solution of the equation satisfying the following initial and boundary conditions is sought

$$\Psi(0,\xi,\eta)=0,\qquad \frac{\partial\Psi}{\partial\eta}(\tau,\xi,1)=\frac{\partial\Psi}{\partial\eta}(\tau,\xi,0)=0. \qquad [23]$$

The solution is represented as a series

$$\Psi(\tau,\xi,\eta)=\sum_{n=1}^{\infty}\frac{u_n(\tau,\xi)}{\|X_n\|^2}\cdot X_n(\eta),\qquad \|X_n\|^2=\int_0^1 X_n^2(\eta)\cdot\eta\,\mathrm{d}\eta. \qquad [24]$$

Here $X_n(\eta)$ are eigenfunctions of the following Sturm-Liouville problem

$$(\eta X_n'(\eta))'+\lambda_n^2\eta X_n(\eta)=0,\qquad X_n'(0)=X_n'(1)=0. \qquad [25]$$

Function $\Psi(\tau,\xi,\eta)$ defined by series [24] is substituted into [22]. Then the L.H.S. and the R.H.S. of [22] are multiplied by $\eta X_n(\eta)$ and are integrated by η within the bounds of from 0 to 1. Taking into account the orthogonality of Eigen-functions $X_n(\eta)$ and boundary conditions [23] we get the following equation for defining functions $u_n(\tau,\xi)$

$$\frac{\partial u_n}{\partial\tau}+\lambda_n^2u_n=-Y\frac{\partial\theta}{\partial\xi}\cdot\int_0^1(\Phi(\eta)-1)\eta X_n(\eta)\,\mathrm{d}\eta. \qquad [26]$$

According to the initial condition for $\Psi(\tau, \xi, \eta)$ function $u_n(\tau, \xi)$ is equal to zero when $\tau = 0$.

$$u_n(\tau, \xi) = 0. \tag{27}$$

The solution for function $u_n(\tau, \xi)$ is

$$\begin{gathered} u_n(\tau, \xi) = -a_n Y \cdot \int_0^\tau \exp\left[-\lambda_n^2(\tau - s)\right] \frac{\partial \theta}{\partial \xi}(s, \xi)\, \mathrm{d}s, \\ a_n = \int_0^1 (\Phi(\eta) - 1)\eta X_n(\eta)\, \mathrm{d}\eta. \end{gathered} \tag{28}$$

Using these equalities we get [24] as

$$\Psi(\tau, \xi, \eta) = -Y \sum_{n=1}^{\infty} \frac{a_n X_n(\eta)}{\|X_n\|^2} \int_0^\tau \exp\left[-\lambda_n^2(\tau - s)\right] \frac{\partial \theta}{\partial \xi}(s, \xi)\, \mathrm{d}s. \tag{29}$$

5.2 *One-dimension model of mass transfer*

The approximate solution found above is used for developing one-dimension equation for mean mixture concentration $\theta(\tau, \xi)$. Equation [29] is substituted into the integral addend in the R.H.S. of [20]. We get

$$\begin{gathered} \frac{\partial \theta}{\partial \tau} + Y \frac{\partial \theta}{\partial \xi} = \frac{\partial^2 \theta}{\partial \xi^2} + 2Y^2 \sum_{n=1}^{\infty} \frac{a_n^2}{\|X_n\|^2} \int_0^\tau \exp\left[-\lambda_n^2(\tau - s)\right] \frac{\partial^2 \theta}{\partial \xi^2}(s, \xi)\, \mathrm{d}s, \\ \theta = \theta(\tau, \xi), \quad \tau > 0, \quad -\infty < a < \xi < b < +\infty. \end{gathered} \tag{30}$$

This equation includes only one unknown function $\theta(\tau, \xi)$ and describes the longitudinal matter transfer in a laminar flow of a newtonian or non-newtonian fluid in a round tube. It can be represented as

$$\begin{aligned} \frac{\partial \theta}{\partial \tau} + Y \frac{\partial \theta}{\partial \xi} = {} & \left(1 + 2Y^2 \sum_{n=1}^{\infty} \frac{a_n^2}{\lambda_n^2 \|X_n\|^2}\right) \frac{\partial^2 \theta}{\partial \xi^2} \\ & -2Y^2 \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_n^2}{\lambda_n^2 \|X_n\|^2} \left(\frac{\partial}{\partial \tau} \int_0^\tau \exp\left[-\lambda_n^2(\tau - s)\right] \frac{\partial^2 \theta}{\partial \xi^2}(s, \xi)\, \mathrm{d}s\right). \end{aligned} \tag{31}$$

This equation is different from the diffusive transfer equations (Taylor 1953; Aris 1956). This difference is in the integral addend in the R.H.S. of [31]. This addend is connected with the delay effects of concentration alignment in the tube cross-section. That is why the integral addend disappears when $\tau \to \infty$ and [31] becomes the diffusion equation of the following kind

$$\frac{\partial \theta}{\partial \tau} + Y \frac{\partial \theta}{\partial \xi} = \frac{K}{D} \frac{\partial^2 \theta}{\partial \xi^2}, \qquad \frac{K}{D} = 1 + 2Y^2 \sum_{n=1}^{\infty} \frac{a_n^2}{\lambda_n^2 \|X_n\|^2}. \tag{32}$$

Here K/D is the dimensionless virtual coefficient. Hence Taylor's diffusion coefficient is equal to

$$\frac{K}{D} = 2Y^2 \sum_{n=1}^{\infty} \frac{a_n^2}{\lambda_n^2 \|X_n\|^2}. \tag{33}$$

When $\tau \to 0$ the diffusion equation follows from [31]

$$\frac{\partial \theta}{\partial \tau} + Y\frac{\partial \theta}{\partial \xi} = \frac{\partial^2 \theta}{\partial \xi^2}, \quad \theta = \theta(\tau, \xi). \tag{34}$$

It follows from [34] that even at $Pe \gg 1$ ($Pe = 2Y$) when Taylor's diffusion coefficient is much larger than the molecular one there exists a range of τ values where the molecular diffusion cannot be neglected. The same result is received in the paper (Ananthakrishnan *et al.* 1965) by a numerical solution of [14].

5.3 *Calculation of* a$_n$

The solution of the Sturm–Liouville problem has the form

$$X_n(\eta) = J_0(\lambda_n \eta). \tag{35}$$

Here $J_0(\lambda_n \eta)$ is Bessel's function of the first power, λ_n is the positive root of equation $J_1(\lambda) = 0$. We get

$$\|X_n\|^2 = \frac{1}{2} J_0^2(\lambda_n). \tag{36}$$

For calculating a_n coefficients the velocity profile must be defined. For a large class of media including the newtonian fluids, the velocity profile can be taken as

$$\Phi(\eta) = \frac{3m+1}{m+1}\left(1 - \eta^{(m+1)/m}\right). \tag{37}$$

Here m is the power factor in the power rheological law. When $m = 1$, the velocity profile is parabolic. Using [35] and [37] a_n can be written as

$$a_n = \frac{3m+1}{m} \cdot \frac{1}{\lambda_n} \int_0^1 J_1(\lambda_n \eta) \cdot \eta^{(m+1)/m} \, d\eta. \tag{38}$$

When $m = 1$, [38] is simplified to

$$a_n = -\frac{4}{\lambda_n^2} J_0(\lambda_n). \tag{39}$$

5.4 *Calculation of virtual coefficient*

Let us take $m = 1$, then

$$\frac{K}{D} = 1 + 64 Y^2 \sum_{n=1}^{\infty} \frac{1}{\lambda_n^6}. \tag{40}$$

It has been shown (Watson 1922) that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^6} = \frac{1}{3072}. \tag{41}$$

Therefore

$$\frac{K}{D} = 1 + \frac{1}{48} Y^2 = 1 + \frac{1}{48} \frac{U_0^2 R^2}{D^2}. \tag{42}$$

The virtual coefficient has precisely this value in the Taylor–Aris model generalising Taylor's model for small values of $Y \sim 1$. In Taylor's equation this coefficient is equal to $\frac{1}{48}Y^2$. At $m \neq 1$, the virtual coefficient is represented as

$$\frac{K}{D} = 1 + G(m)Y^2,$$

$$G(m) = 4\left(\frac{3m+1}{m}\right)^2 \cdot \sum_{n=1}^{\infty} \frac{b_n{}^2}{\lambda_n{}^2 J_0{}^2(\lambda_n)}, \qquad [43]$$

$$b_n = \int_0^1 J_1(\lambda_n \eta) \cdot \eta^{(m+1)/m}\, \mathrm{d}\eta.$$

The value of factor $G(m)$ is shown in table 1 for different values of m.

Table 1.

m	5	2	1	0.9	0.7	0.5	0.3	0.1
$G(m)$	0.0303	0.0260	0.0206	0.0196	0.0171	0.0143	0.0094	0.0026

It is evident from the table that the value of factor $G(m)$ in [43] quickly decreases with decreasing of m. It means that the sizes of mixture area in a pseudo-plastic fluid flow will be less than these sizes in newtonian and dilatant fluid flows.

6. MASS TRANSFER MODEL IN A NEWTONIAN FLUID

Equation [30] is written for the newtonian fluid flow $m = 1$. We get

$$\frac{\partial \theta}{\partial \tau} + Y\frac{\partial \theta}{\partial \xi} = \frac{\partial^2 \theta}{\partial \xi^2} + Y^2 \int_0^{\tau} W(\tau - s)\frac{\partial^2 \theta}{\partial \xi^2}(s, \xi)\, \mathrm{d}s,$$

$$\theta = \theta(\tau, \xi), \tau > 0, \quad -\infty < a < \xi < b < +\infty, \qquad [44]$$

$$W(\tau) = 64 \sum_{n=1}^{\infty} \frac{1}{\lambda_n^4} \exp(-\lambda_n{}^2 \tau).$$

This equation generalises the well-known Taylor and Taylor–Aris models in the sense that these models are obtained from [44] when $\tau \to \infty$. It is interesting to note that [44] has no virtual coefficient.

When function $\theta(\tau, \xi)$ satisfies [44], its dispersion changes by the law

$$\sigma^2 = 2\left[\left(1 + \frac{1}{48}Y^2\right)\tau - 64Y^2 \sum_{n=1}^{\infty} \frac{1 - \exp(-\lambda_n{}^2\tau)}{\lambda_n{}^8}\right]. \qquad [45]$$

We took $\sigma(0) = 0$.

Chatwin (1970) got a similar law for dispersion depending on time by another method using some particular assumptions.

As was done by Taylor, the length of the mixture area is defined in the following way

$$l_s = 2.56\sigma. \qquad [46]$$

Using this formula and the expression for dispersion [45], the calculations of the mixture area length have been made for the first series of experiments (Bailey & Gogarty 1962). The results of the calculations are shown graphically in figure 1. A dotted line in the same figure shows the results of calculations on the basis of Taylor's model with the linear law of dispersion and the experimental data. The comparison of the curves and experimental data gives credence to the suggested theory. That is why model [44] can be used for describing the process when $\tau < 1$, where Taylor's theory is not true.

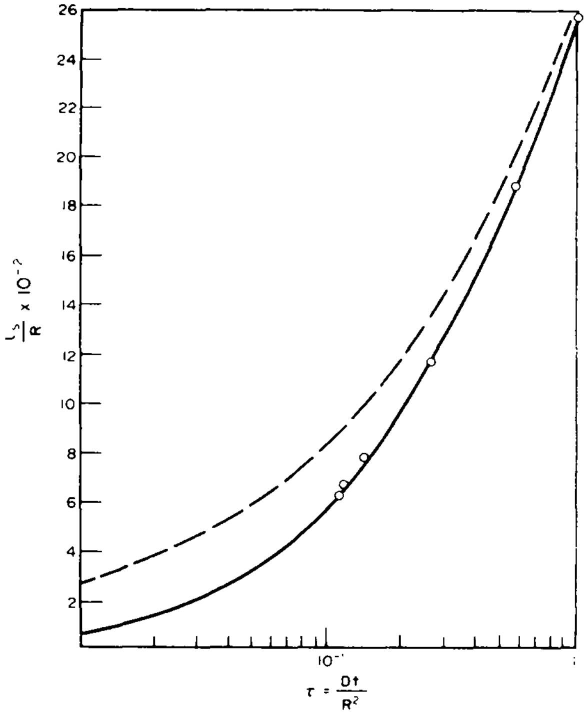

Figure 1. The continuous line shows the calculations on the basis of the author's theory. The dotted line shows the calculations on the basis of Taylor's theory. O—shows the experimental data (Bailey & Gogarty 1962).

The fact that the given model is close to Taylor's model when $\tau \to \infty$ enables us to estimate the limits of application of the latter. To do that for parameters Pe and τ the equation for the velocity of dispersion change is used.

$$\frac{d\sigma^2}{d\tau} = 2\left[1 + 64Y^2 \sum_{n=1}^{\infty} \frac{1 - \exp(-\lambda_n^2 \tau)}{\lambda_n^6}\right]. \qquad [47]$$

The first addend in square brackets of [47] describes the molecular diffusion contribution into the longitudinal transfer, and the second addend, the contribution of Taylor diffusion. The contributions of both diffusive processes would be the same if the condition

$$64Y^2 \sum_{n=1}^{\infty} \frac{1 - \exp(-\lambda_n^2 \tau)}{\lambda_n^6} \sim 1 \qquad [48]$$

is satisfied.

For the small typical time moments, when $\lambda_1^2 \tau \ll 1$, this condition can be rewritten as

$$64Y^2 \tau \sum_{n=1}^{\infty} \frac{1}{\lambda_n^4} \sim 1. \qquad [49]$$

It follows from [49] that

$$Pe \sim 3.66 \cdot \tau^{-1/2}, \qquad Pe = 2Y. \qquad [50]$$

The same estimate can be obtained when mixture areas formed by the admixture particles moving and formed by the molecular diffusion along the axis are taken to have the same lengths. We get

$$\frac{2U_0 t}{3.62\sqrt{Dt}} \sim 1 \Rightarrow Pe \sim 3.62\tau^{-1/2}. \qquad [51]$$

To make the contribution of Taylor diffusion the main and the molecular one neglectable the following inequality should be taken

$$64Y^2\sum_{n=1}^{\infty}\frac{1-\exp(-\lambda_n^2\tau)}{\lambda_n^6}\gg 1. \qquad [52]$$

At $\tau\to\infty$, Taylor's condition $Pe\gg 13.8$ follows from this inequality.

Taylor (1954) had taken the axial molecular diffusion be neglected under this condition. But in fact the problem is more complex. It has been shown above that at any Pe number there are such time values τ for which the molecular transfer plays the main part.

7. LAMINAR FLOW IN A FLAT CHANNEL

In this case the symmetry factor p is equal to 0. Under this condition the transfer model developing of which is similar to that of [30], has the form

$$\frac{\partial\theta}{\partial\tau}+Y\frac{\partial\theta}{\partial\xi}=\frac{\partial^2\theta}{\partial\xi^2}+\frac{1}{2}Y^2\cdot\sum_{n=1}^{\infty}\frac{a_n^2}{\|X_n\|^2}\int_0^{\tau}\exp[-\lambda_n^2(\tau-s)]\frac{\partial^2\theta}{\partial\xi^2}(s,\xi)\,\mathrm{d}s \qquad [53]$$

$$\theta=\theta(\tau,\xi),\quad \tau>0,\quad -\infty<a<\xi<b<+\infty.$$

Here

$$\tau=\frac{Dt}{h^2},\qquad \xi=\frac{x}{h},\quad \eta=\frac{r}{h},\quad Y=\frac{U_0h}{D}$$

$2h$ is the channel height.

The eigenfunctions $X_n(\eta)$ satisfy the following Sturm-Liouville problem

$$X_n''(\eta)+\lambda_n^2X_n(\eta)=0,\qquad X_n'(-1)=X_n'(1)=0. \qquad [54]$$

It is simply shown that

$$X_n(\eta)=\cos\lambda_n(1-\eta),\qquad \lambda_n=\frac{\pi}{2}n,\quad n=1,2,3\ldots$$

$$\|X_n\|^2=\int_{-1}^{1}X_n^2(\eta)\,\mathrm{d}\eta=1. \qquad [55]$$

a_n coefficients are equal to

$$a_n=\int_{-1}^{1}(\Phi(\eta)-1)X_n(\eta)\,\mathrm{d}\eta. \qquad [56]$$

The velocity profile in the newtonian fluid flow in a flat split has the form

$$\Phi(\eta)=\frac{3}{2}(1-\eta^2),\qquad -1\leqslant\eta\leqslant 1. \qquad [57]$$

For this velocity profile we have

$$a_n=-\frac{3}{\lambda_n^2}(1+(-1)^n). \qquad [58]$$

Hence it follows that

$$a_{2k-1} = 0, \quad a_{2k} = -\frac{6}{\lambda_{2k}^2} = -\frac{6}{\pi^2 k^2}, \quad k = 1, 2, 3 \ldots \qquad [59]$$

Taking into account [55] and [59], [53] is represented as

$$\frac{\partial\theta}{\partial\tau} + Y\frac{\partial\theta}{\partial\xi} = \frac{\partial^2\theta}{\partial\xi^2} + \frac{18}{\pi^4} Y^2 \sum_{n=1}^{\infty} \frac{1}{k^4} \int_0^{\tau} \exp\left[-k^2\pi^2(\tau - s)\right] \frac{\partial^2\theta}{\partial\xi^2}(s, \xi)\,\mathrm{d}s \qquad [60]$$

$$\theta = \theta(\tau, \xi), \quad \tau > 0, \quad -\infty < a < \xi < b < +\infty.$$

At $\tau \to \infty$, The solutions of this equation are asymptotically close to the solutions of the diffusion equation of the following kind

$$\frac{\partial\theta}{\partial\tau} + Y\frac{\partial\theta}{\partial\xi} = \left(1 + \frac{18}{\pi^6} Y^2 \sum_{k=1}^{\infty} \frac{1}{k^6}\right) \cdot \frac{\partial^2\theta}{\partial\xi^2}. \qquad [61]$$

The second addend in round brackets is equal to ζ-function for the argument equal to six.

$$\sum_{k=1}^{\infty} \frac{1}{k^6} = \zeta(6) = \frac{\pi^6}{945}. \qquad [62]$$

The equality of the virtual coefficient value for flow in a flat split to

$$\frac{K}{D} = 1 + \frac{2}{105} Y^2, \qquad [63]$$

follows from [62].

The same expression is represented by Philip (1963).

The dispersion of function $\theta(\tau, \xi)$ for [60] is equal to

$$\sigma^2 = 2\left[\left(1 + \frac{2}{105} Y^2\right)\tau - \frac{18}{\pi^6} Y^2 \sum_{k=1}^{\infty} \frac{1 - \exp(-k^2\pi^2\tau)}{k^8}\right]. \qquad [64]$$

8. APPROXIMATION OF [44]

It has been shown above that the solutions of [44] are asymptotically close to the solutions of diffusive equations [32] and [34], when $\tau \to \infty$ and $\tau \to 0$ respectively. Another approximation will be shown below for the same [44].

With this aim in view the movable coordinate system $z = \xi - \tau$ is considered and function $W(\tau)$ is approximated in the following way.

$$W(\tau) \approx \frac{64}{\lambda^4} \exp(-\lambda_1^2 \tau).$$

In this case [44] without the addend $\partial^2\theta/\partial\xi^2$ will have the form

$$\frac{\partial\theta}{\partial\tau} = \frac{64}{\lambda_1^4} Y^2 \int_0^{\tau} \exp\left[-\lambda_1^2(\tau - s)\right] \frac{\partial^2\theta}{\partial z^2}(s, z)\,\mathrm{d}s. \qquad [65]$$

This equation can be rewritten as

$$\frac{\partial^2\theta}{\partial\tau^2}+\lambda_1{}^2\frac{\partial\theta}{\partial\tau}=\frac{64}{\lambda_1^4}Y^2\frac{\partial^2\theta}{\partial z^2}, \qquad (\lambda_1=3.83). \tag{66}$$

Equation [66] is known as a telegraph equation. The final velocity of the admixture spread equal to $0.75U_{\max}$ follows from the solution of this equation. This result has a physical meaning because the velocity of the admixture particles cannot be larger than the velocity on a tube axis. The indefinite velocity of disturbance spreading follows from Taylor's model. The telegraph equation has already been applied to describing the heat and matter transfer (Goldstein 1951). This equation is free from those paradoxes which follow from the solution of the heat conductivity type equation.

9. TURBULENT FLOW IN A ROUND TUBE

In this case the local diffusion coefficient $\epsilon(r)$ is different from the molecular one and is changing along a tube cross-section. It is equal to

$$\epsilon(r)=D+D_t(r). \tag{67}$$

Here $D_t(r)$ is coefficient of the turbulent diffusion.

The dimensionless variables are introduced

$$\tau=\frac{\epsilon_0 t}{R^2}, \qquad \xi=\frac{x}{R}, \qquad \eta=\frac{r}{R}. \tag{68}$$

Here ϵ_0 is the characteristic value of $\epsilon(r)$.

When an axial diffusion which is an order less than Taylor's one in a turbulent flow, is not taken into account, the transfer equation can be represented as

$$\begin{gathered}\frac{\partial\theta}{\partial\tau}zffl Y_t\frac{\partial\theta}{\partial\xi}=2Y_t^2\sum_{n=1}^{\infty}\frac{a_n{}^2}{\|X_n\|^2}\cdot\int_0^{\tau}\exp\left[-\lambda_n{}^2(\tau-s)\right]\frac{\partial^2\theta}{\partial\xi^2}(s,\xi)\,\mathrm{d}s,\\ Y_t=\frac{U_0R}{\epsilon_0}, \quad \theta=\theta(\tau,\xi), \quad \tau>0, \quad -\infty<a<\xi<b<+\infty,\end{gathered} \tag{69}$$

for function $\theta(\tau,\xi)$.

a_n coefficients are calculated by means of [28]. However eigenfunctions $X_n(\eta)$ in [28] satisfy the following Sturm–Liouville problem

$$\begin{gathered}(\eta\epsilon_*(\eta)X'_n(\eta))'+\lambda_n{}^2\eta X_n(\eta)=0,\\ X'_n(0)=X'_n(1)=0, \quad \epsilon_*=\frac{\epsilon(\eta)}{\epsilon_0}.\end{gathered} \tag{70}$$

9.1. *Calculation of a_n coefficients*

Using [70] a_n is represented as follows

$$a_n=-\frac{1}{\lambda_n{}^2}\int_0^1(\Phi(\eta)-1)(\eta\epsilon_*(\eta)X'_n(\eta))'\,\mathrm{d}\eta=\frac{1}{\lambda_n{}^2}\int_0^1\eta\epsilon_*(\eta)X'_n(\eta)\Phi'(\eta)\,\mathrm{d}\eta. \tag{71}$$

Coefficient D_t of turbulent diffusion is calculated by means of formula

$$D_t(\eta) = -\frac{u_*^2 R\eta}{U_0}(\Phi'(\eta))^{-1}, \qquad [72]$$

u_* is the friction velocity. Hence

$$\epsilon_* = \frac{D}{\epsilon_0} - \frac{u_*^2 R\eta}{U_0\epsilon_0}(\Phi'(\eta))^{-1}. \qquad [73]$$

This expression is substituted into [71]. We get

$$a_n = \frac{D}{\lambda_n^2\epsilon_0}\int_0^1 \eta X_n'(\eta)\Phi'(\eta)\,d\eta - \frac{u_*^2 R}{U_0\epsilon_0}X_n(1). \qquad [74]$$

The value of D/ϵ_0 is estimated. The typical value of the transfer coefficient ϵ_0 can be taken as $\epsilon_0 \sim u_*R$. Then

$$\frac{D}{\epsilon_0} \sim \frac{D}{u_*R} \sim \frac{1}{Sc}\cdot\frac{1}{Re_*}, \quad Sc = \frac{\nu}{D}, \quad Re_* = \frac{u_*R}{\nu},$$

ν is the kinematic viscosity.

The first addend in [74] can be neglected for media where $Sc \gg 1$. Then

$$a_n = -\frac{u_*^2 R}{\epsilon_0 U_0}X_n(1). \qquad [75]$$

Hence the dependence of K/ϵ_0 virtual coefficient defined by [33] on $X_n(1)$ only and its independence of the velocity profile kind follows.

10. FLOW IN AN OPEN CHANNEL

The local transfer equation keeps the kind of [14] in this case if $p = 0$. The dimensionless variables are introduced

$$\tau = \frac{\epsilon_0 t}{H^2}, \quad \xi = \frac{x}{H}; \quad \eta = \frac{r}{H}, \quad Y_* = \frac{u_*H}{\epsilon_0}.$$

Here H is the depth of the channel.

The dimensionless velocity profile is taken to be equal to

$$\Phi(\eta) = \frac{U}{u_*} + \frac{1}{\chi}[\sqrt{(1-\eta)} + \ln(1-\sqrt{(1-\eta)})]. \qquad [76]$$

Here U is velocity on a free surface, $\chi = 0.3$.

According to [72] the turbulent transfer coefficient is defined as

$$D_t(\eta) = 2\chi u_* H(1-\eta)\cdot(1-\sqrt{1-\eta}). \qquad [77]$$

The typical value of ϵ_0 diffusion coefficient is taken to be equal to the mean value of $\epsilon(\eta)$. Then

$$\epsilon_0 = \int_0^1 \epsilon(\eta)\,\mathrm{d}\eta = D + 0.06u_*H. \tag{78}$$

The one-dimensional model for function $\theta(\tau, \xi)$ has the form

$$\begin{gathered}\frac{\partial\theta}{\partial\tau} + Y_*\frac{\partial\theta}{\partial\xi} = Y_*^2\sum_{n=1}^{\infty}\frac{a_n^2}{\|X_n\|^2}\cdot\int_0^{\tau}\exp\left[-\lambda_n^2(\tau - s)\right]\frac{\partial^2\theta}{\partial\xi^2}(s, \xi)\,\mathrm{d}s.\\ \theta = \theta(\tau, \xi), \quad \tau > 0, \quad -\infty < a < \xi < b < +\infty.\end{gathered} \tag{79}$$

a_n coefficient is equal to

$$a_n = \frac{D}{\lambda_n^2\epsilon_0}\int_0^1 \Phi'(\eta)X_n'(\eta)\,\mathrm{d}\eta - \frac{u_*H}{\lambda_n^2\epsilon_0}\cdot X_n(0). \tag{80}$$

Eigen-functions $X_n(\eta)$ satisfy the following Sturm–Liouville problem

$$\begin{gathered}(\epsilon_*(\eta)X_n'(\eta))' + \lambda_n^2X_n(\eta) = 0, \qquad X_n'(0) = X_n'(1) = 0,\\ \|X_n\|^2 = \int_0^1 X_n^2(\eta)\,\mathrm{d}\eta.\end{gathered} \tag{81}$$

The difficulty in solving [81] is connected with the dependence of radial diffusion coefficient $\epsilon(\eta)$ on η. The experiments enable us to estimate the nature of this dependence. $\epsilon(\eta)$ coefficient is almost constant in a core of turbulent flow which occupies the main part of tube cross-section. Such a dependence of ϵ on η gives the reason to take the value of $\epsilon(\eta)$ to be constant and equal to the mean value of ϵ_0 as the first approximation when solving Sturm–Liouville problem. In this case we have

$$\begin{gathered}X_n(\eta) = \cos n\pi\eta, \qquad \lambda_n = n\pi, \qquad \|X_n\|^2 = \frac{1}{2}.\\ n = 1, 2, 3\ldots\end{gathered} \tag{82}$$

Under this condition a_n coefficient is equal to

$$a_n = \frac{u_*H}{\lambda_n^2\epsilon_0}\left(1 + \frac{\lambda_nD}{u_*H}\int_0^1 \Phi'(\eta)\sin\lambda_n\eta\,\mathrm{d}\eta\right). \tag{83}$$

It is simply shown that

$$\int_0^1 \Phi'(\eta)\cdot\sin\lambda_n\eta\cdot\mathrm{d}\eta = \frac{1}{\chi}\int_0^1 \frac{\sin\pi\eta}{1 - \sqrt{1 - \eta}}\,\mathrm{d}\eta \approx 3.1.$$

10.1. *Calculation of virtual coefficient*

The virtual coefficient is equal to

$$\frac{K}{\epsilon_0} = Y_*^2\sum_{n=1}^{\infty}\frac{a_n^2}{\lambda_n^2\|X_n\|^2}. \tag{84}$$

for model [79].

Using the expressions found above we get

$$\frac{K}{\epsilon_0} = 2Y_*^2 \cdot \sum_{n=1}^{\infty} \frac{u_*^2 H^2}{\lambda_n^6 \epsilon_0^2} \left(1 + \frac{3.1 \cdot \lambda_n D}{u_* \cdot H}\right)^2. \tag{85}$$

Because of the fast increase of λ_n eigenvalues it is possible to write down approximately the following

$$\frac{K}{\epsilon_0} \approx 2Y_*^2 \frac{u_*^2 H^2}{\pi^6 \cdot \epsilon_0^2} \left(1 + \frac{3.1 \pi D}{u_* H}\right)^2.$$

As a result of simple calculations we get the following expression

$$\frac{K}{u_* H} = 9.8 \left(1 + \frac{9.7}{Sc \cdot Re_*}\right)^2 \cdot \left(1 + \frac{12.5}{Sc \cdot Re_*}\right)^{-3}, \tag{86}$$

$$Re_* = u_* H / \nu$$

If the molecular transfer is neglected we get

$$\frac{K}{u_* H} = 9.8 \tag{87}$$

10.2. *Admixture dispersion*

An admixture dispersion for model [79] is subject to the following law

$$\sigma^2 = 2Y_*^2 \sum_{n=1}^{\infty} \frac{a_n^2}{\lambda_n^2 \|X_n\|^2} \left(\tau - \frac{1 - \exp(-\lambda_n^2 \tau)}{\lambda_n^2}\right). \tag{88}$$

Because of the fast increase of λ_n eigenvalues, [88] are represented as follows

$$\sigma^2 \approx 2Y_*^2 \sum_{n=1}^{\infty} \frac{a_n^2}{\lambda_n^2 \|X_n\|^2} \left(\tau - \frac{1 - \exp(-\lambda_1^2 \tau)}{\lambda_1^2}\right) \approx 2 \frac{K}{\epsilon_0} \left(\tau - \frac{1 - \exp(-\pi^2 \cdot \tau)}{\pi^2}\right). \tag{89}$$

It has been shown by Taylor (1921) that the admixture dispersion could be found from the following equality

$$\frac{d\sigma^2}{dt'} = 2 \int_0^{t'} R(s)\, ds. \tag{90}$$

Here $R(t')$ is the coefficient of Lagrange correlation. When $t' \to \infty$ the dispersion increases proportionally to time t'. When t' is small the dispersion is proportional to the square of time t'.

In a previous paper (Sullivan 1971), function $R(t')$ is taken as

$$R(t') = \frac{b \cdot A}{u_* H} \exp(-bt'), \qquad t' = \frac{u_* t}{H}. \tag{91}$$

Here b is constant.

For function $R(t')$ dispersion is equal to

$$\sigma^2 = 2 \frac{A}{u_* H} \left(t' - \frac{1 - \exp(-bt')}{b}\right). \tag{92}$$

Sullivan has given the following values for b and A: when $Re_* = 500$, $b = 0.536$, $A = 11u_*H$.

It has been simply noted that the dependences [89] and [92] are the same. Equation [89] is rewritten for time t'. We get

$$\tau = \frac{\epsilon_0 t}{H^2} = \frac{\epsilon_0}{u_*H}t' = \left(\frac{D}{u_*H} + 0.06\right)t' = \left(\frac{1}{ScRe_*} + 0.06\right)t'.$$

This relationship is substituted into [89].

$$\sigma^2 = 2\frac{K}{u_*H}\left(t' - \frac{1 - \exp(-bt')}{b}\right), \quad b = \pi^2\left(\frac{1}{ScRe_*} + 0.06\right). \qquad [93]$$

For $Re_* = 500$ the following can be taken $b \approx \pi^2 \cdot 0.06 = 0.59$, $K = 9.8_*H$. These values calculated on the base of the given theory, are close to those shown by Sullivan (1971).

11. EFFECT OF THE INITIAL CONDITIONS

So far when deriving a one-dimension equation for function $\theta(\tau, \xi)$ an agreement of the initial conditions for this function with local concentration $C(\tau, \xi, \eta)$ was assumed, that is, $C(0, \xi, \eta) = \theta_0(\xi)$. In this case the initial condition has the form $u_n(0, \xi) = 0$ when defining functions $u_n(\tau, \xi)$. If the initial distribution for $C(\tau, \xi, \eta)$ depends on variable η then $u_n(0, \xi) = u_n^0(\xi)$. Subject to this condition the solution of [26] is found as

$$u_n(\tau, \xi) = u_n^0(\xi)\exp(-\lambda_n^2\tau) - a_nY\int_0^\tau \exp[-\lambda_n^2(\tau - s)]\frac{\partial\theta}{\partial\xi}(s, \xi)\,ds. \qquad [94]$$

Hence the one-dimension equation for $\theta(\tau, \xi)$ has the form

$$\frac{\partial\theta}{\partial\tau} + Y\frac{\partial\theta}{\partial\xi} + 2Y\sum_{n=1}^{\infty}\frac{a_n}{\|X_n\|^2}\frac{\partial u_n^0}{\partial\xi}\cdot\exp(-\lambda_n^2\tau)$$
$$= \frac{\partial^2\theta}{\partial\xi^2} + 2Y^2\sum_{n=1}^{\infty}\frac{a_n^2}{\|X_n\|^2}\cdot\int_0^\tau \exp[-\lambda_n^2(\tau - s)]\frac{\partial^2\theta}{\partial\xi^2}(s, \xi)\,ds. \qquad [95]$$

An additional addend occurs in this equation which is absent in [30]. This addend decreases with time as $\exp(-\lambda_n^2\tau)$. That is why the form of the asymptotic model is the same as [32] when $\tau \to \infty$. It will be noted that in Taylor's theory it is impossible to take into account the disagreement of the initial conditions. At the same time the consideration of such a disagreement seems to be important when analysing the experimental data. For example when a matter portion is injected into the flow it is difficult to reach the uniform distribution of matter along the tube cross-section. That is the reason for the local concentration depending on two space variables $C(0, \xi, \eta) = C_0(\xi, \eta)$ at moment $\tau = 0$. If function $C_0(\xi, \eta)$ is defined then

$$u_n^0(\xi) = \int_0^1 C_0(\xi, \eta)X_n(\eta)\cdot\eta\,d\eta. \qquad [96]$$

12. RADIOACTIVE ADMIXTURE DISPERSION IN FLOW IN A TUBE

Description of radioactive admixture concentration is connected with consideration of the radioactive decay phenomenon. That is the reason for the one-dimension model of diffusion being different from [30]. This difference is not only in the additional addend accounting the radioactive decay but in the dependence of virtual coefficient on the decay constant. The one-dimension model in this case has the following form case has the following form

$$\frac{\partial\theta}{\partial\tau}+Y\frac{\partial\theta}{\partial\xi}+\Gamma\theta=\frac{\partial^2\theta}{\partial\xi^2}+Y^2\sum_{n=1}^{\infty}\frac{a_n{}^2}{\|X_n\|^2}\int_0^{\tau}\exp\left[-(\lambda_n{}^2+\Gamma)(\tau-s)\right]\frac{\partial^2\theta}{\partial\xi^2}(s,\xi)\,\mathrm{d}s,\quad \Gamma=\frac{\beta R^2}{D}$$

$$\theta=\theta(\tau,\xi),\quad \tau>0,\quad -\infty<a<\xi<b<+\infty \qquad [97]$$

(β half-life period).

The virtual coefficient is equal to

$$\frac{K}{D}=1+64Y^2\sum_{n=1}^{\infty}\frac{1}{\lambda_n{}^6(1+\lambda_n{}^{-2}\cdot\Gamma)}. \qquad [98]$$

Hence this coefficient is less than Taylor's.

When $\tau\to\infty$ the admixture dispersion is equal to

$$\sigma^2=\frac{2K}{\Gamma D}[1-\exp(-\Gamma\tau)]. \qquad [99]$$

The dispersion change law differs from the linear law as it takes place for non-radioactive admixture.

CONCLUSIONS

The suggested theory can be generalised on the description of newtonian and non-newtonian media mixing in more general cases, in capillary-porous systems, in particular (Nikolaevskii *et al.* 1968; Maron 1974). Such a generalisation presents considerable interest for a series of chemical technology processes and for the chromatography theory. An agreement of the mass transfer theory in laminar flow in a round tube with the experiments (Bailey & Gogarty 1962) enables us to recommend it when $\tau \ll 1$. Developing of mass transfer models in turbulent flow is connected with defining function $\epsilon(\eta)$ and solving the corresponding Sturm–Liouville problem. For the models shown the third distribution moment defining the asymmetry is equal to zero. To specify this result it is necessary to find the next approximate solution of [21] and to apply it to defining model [30].

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